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A generalized Ginzburg–Landau functional for systems with correlation

U Lindner
Sektion Physik, Karl-Marx-Universität, 7010 Leipzig, Germany

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Abstract. Using the theory of effective action a generalized Ginzburg–Landau functional for systems with correlation is derived. The Bardeen–Cooper–Schrieffer theory is obtained as a special case.

1. Introduction

The discovery of high-temperature superconductivity in La–(Ba, Sr)–Cu–O [1] and in other oxide systems has led in recent years to further extensive studies of the Hubbard model [2] to examine whether this model is suitable for explaining the magnetic and superconducting behaviour of these systems (see, e.g., [3]). Since apart from special cases [4] the single-band Hubbard model is not exactly solvable in dimensions higher than one, it is necessary to use approximations.

In this paper we apply the method of effective action (see, e.g., [5, 6]) and obtain a Ginzburg–Landau functional which depends on two real and two complex fields and which makes it possible to discuss the phase transition of our model. The Ginzburg–Landau functional for Bardeen–Cooper–Schrieffer (BCS) superconductors is obtained as a special case.

2. Model

The s-band Hubbard model in the field representation is given by

$$\hat{H} = \hat{H}_0 + \hat{H}_1$$

with

$$\hat{H}_0 \equiv \sum_{\sigma=\uparrow,\downarrow} \int_V \mathrm{d}^3r\, \hat{\psi}_\sigma^+(r) \left(\frac{-\nabla^2}{2m} - \mu \right) \hat{\psi}_\sigma(r) \tag{1}$$

and

$$\hat{H}_1 = U \int_V \mathrm{d}^3r\, \hat{\psi}_\uparrow^+(r) \hat{\psi}_\downarrow^+(r) \hat{\psi}_\downarrow(r) \hat{\psi}_\uparrow(r)$$

where $\hat{\psi}_\sigma^+(r)$ and $\hat{\psi}_\sigma(r)$ are the creation and annihilation operators for an electron with spin σ at the position r, $-\nabla^2/2m$ is the kinetic energy operator of an electron ($\hbar = 1$) and

μ is the chemical potential. V denotes the periodicity volume; this means that we use periodical boundary conditions. $\hat{H}_1$ describes the correlation, i.e. U is the repulsion energy between electrons with spin $\uparrow$ and $\downarrow$ at the same position. If we use instead of U an attractive coupling constant $-g<0$ between $\uparrow$ and $\downarrow$ spin electrons we obtain the BCS model [6].

The interaction with an electromagnetic field is introduced in a gauge-invariant way by minimal substitution ($c=1$):

$$\nabla \rightarrow \nabla - \mathrm{i}e\boldsymbol{A} \tag{2}$$

and the vector potential $\boldsymbol{A}(\boldsymbol{r})$ is treated as an external source. (The charge of an electron is denoted by e.) Then the Hamiltonian operator

$$\begin{aligned}\hat{H} = \sum_{\sigma=\uparrow,\downarrow} \int_V \mathrm{d}^3r\, \hat{\psi}_\sigma^+(\boldsymbol{r}) \left(-\frac{1}{2m}(\nabla - \mathrm{i}e\boldsymbol{A})^2 - \mu\right) \hat{\psi}_\sigma(\boldsymbol{r}) \\ + U \int_V \mathrm{d}^3r\, \hat{\psi}_\uparrow^+(\boldsymbol{r})\, \hat{\psi}_\downarrow^+(\boldsymbol{r})\, \hat{\psi}_\downarrow(\boldsymbol{r})\, \hat{\psi}_\uparrow(\boldsymbol{r})\end{aligned} \tag{3}$$

is invariant under the gauge transformation

$$\begin{aligned}\hat{\psi}_\sigma(\boldsymbol{r}) &\rightarrow \exp(\mathrm{i}e\Lambda(\boldsymbol{r}))\, \hat{\psi}_\sigma(\boldsymbol{r}) = \hat{\psi}'_\sigma(\boldsymbol{r}) \\ \boldsymbol{A}(\boldsymbol{r}) &\rightarrow \boldsymbol{A}(\boldsymbol{r}) + \nabla\Lambda(\boldsymbol{r}) = \boldsymbol{A}'_\sigma(\boldsymbol{r}).\end{aligned} \tag{4}$$

This means that $\hat{\psi}'_\sigma$ and $\boldsymbol{A}'$ have the same equation of motion as $\hat{\psi}_\sigma$ and $\boldsymbol{A}$. (The phase $\Lambda(\boldsymbol{r})$ does not depend on the spin because of the structure of $\hat{H}_0$.)

3. Derivation of a generalized Ginzburg–Landau functional

Using the holomorphic path integrals [7] and Wick rotation the partition sum is given by

$$Z = \operatorname{tr}\{\exp(-\beta\hat{H})\} = \int_{\substack{\psi(0)=-\psi(\beta)\\ \bar{\psi}(0)=-\bar{\psi}(\beta)}} \mathfrak{D}\bar{\psi}\, \mathfrak{D}\psi \exp(-S[\{\bar{\psi},\psi\}]) \tag{5}$$

with $\beta = 1/k_B T$ and

$$\begin{aligned}\exp(-S) = \exp\left(-\int_0^\beta \mathrm{d}\tau \sum_{\sigma=\uparrow,\downarrow} \int_V \mathrm{d}^3r\, \bar{\psi}_\sigma(\boldsymbol{r},\tau)\left(\frac{\partial}{\partial\tau} - \frac{1}{2m}(\nabla - \mathrm{i}e\boldsymbol{A})^2 - \mu\right)\psi_\sigma(\boldsymbol{r},\tau)\right) \\ \times \exp\left(-U\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3r\, \bar{\psi}_\uparrow(\boldsymbol{r},\tau)\, \bar{\psi}_\downarrow(\boldsymbol{r},\tau)\, \psi_\downarrow(\boldsymbol{r},\tau)\, \psi_\uparrow(\boldsymbol{r},\tau)\right)\end{aligned} \tag{6}$$

where ψ and $\bar{\psi}$ are the Grassmann variables. The four-fermion interaction term in (6) can be expressed in terms of complex scalar fields (the Hubbard–Stratonovich transformation):

$$\begin{aligned}\exp\left(-\int_0^\beta \mathrm{d}\tau\, H_1[\{\bar{\psi},\psi\}]\right) = \int \prod_{i=1}^4 \mathfrak{D}\varphi_i^*\, \mathfrak{D}\varphi_i \frac{1}{C_i} \\ \times \exp\left[-(\gamma_1)^2 \int \mathrm{d}^4y\, \varphi_1^*\varphi_1 + (-\lambda_1 U)^{1/2}\gamma_1 \int \mathrm{d}^4x (\bar{\psi}_\uparrow\bar{\psi}_\downarrow\varphi_1 + \psi_\downarrow\psi_\uparrow\varphi_1^*)\right.\end{aligned}$$

$$-(\gamma_2)^2\int \mathrm{d}^4x\,\varphi_2^*\varphi_2+(\lambda_2 U)^{1/2}\,\gamma_2\int \mathrm{d}^4x\,(\bar\psi_\uparrow\psi_\downarrow\varphi_2+\bar\psi_\downarrow\psi_\uparrow\varphi_2^*)$$

$$-(\gamma_3)^2\int \mathrm{d}^4x\,\varphi_3^*\varphi_3+\left(\frac{-\lambda_3}{2}U\right)^{1/2}$$

$$\times\,\gamma_3\int \mathrm{d}^4x\,[(\bar\psi_\uparrow\psi_\uparrow+\bar\psi_\downarrow\psi_\downarrow)\varphi_3+(\bar\psi_\uparrow\psi_\uparrow+\bar\psi_\downarrow\psi_\downarrow)\varphi_3^*]$$

$$-(\gamma_4)^2\int \mathrm{d}^4x\,\varphi_4^*\varphi_4+\left(\frac{\lambda_4}{2}U\right)^{1/2}$$

$$\times\,\gamma_4\int \mathrm{d}^4x\,[(\bar\psi_\uparrow\psi_\uparrow-\bar\psi_\downarrow\psi_\downarrow)\varphi_4+(\bar\psi_\uparrow\psi_\uparrow-\bar\psi_\downarrow\psi_\downarrow)\varphi_4^*]\Big] \tag{7}$$

where we use the notation

$$\int \mathrm{d}^4x=\int_0^\beta \mathrm{d}\tau\int_V \mathrm{d}^3\boldsymbol{r} \tag{8}$$

and where the constants λ_i, $i=1,\ldots,4$, fulfil the condition $\Sigma_{i=1}^4\lambda_i=1$. The constants λ_i are a consequence of the arbitrariness of the decomposition of the four-fermion term. (Further φ-fields do not occur because terms with $\bar\psi_\uparrow\bar\psi_\uparrow$ and $\bar\psi_\downarrow\bar\psi_\downarrow$ are zero.) The fields φ_3 and φ_4 are real, obviously. The fields φ_1, φ_1^* describe a Cooper pair, the fields φ_2^*, φ_2 correspond to the transverse components S^+, S^- of the spin density, φ_3 denotes the density of electrons and φ_4 is the longitudinal component S^z of the spin density. The normalization constants C_i are given by

$$C_i=\int \mathfrak{D}\varphi_i^*\,\mathfrak{D}\varphi_i\exp\left(-(\gamma_i)^2\int \mathrm{d}^4x\,\varphi_i^*\varphi_i\right)\qquad i=1,\ldots,4 \tag{9}$$

where we need the constants γ_i to ensure the correct dimension of the auxiliary fields. Of course, the physical results have to be independent of these constants.

Now we introduce sources for the auxiliary fields in the usual way and obtain

$$Z[\{j,j^*\}]=\int \mathfrak{D}\bar\psi_\uparrow\,\mathfrak{D}\bar\psi_\downarrow\,\mathfrak{D}\psi_\uparrow\,\mathfrak{D}\psi_\downarrow\prod_{i=1}^4\mathfrak{D}\varphi_i^*\,\mathfrak{D}\varphi_i\,\frac{1}{C_i}$$

$$\times\exp\left[-\int \mathrm{d}^4y\sum_{\sigma=\uparrow,\downarrow}\bar\psi_\sigma\left(\frac{\partial}{\partial\tau}-\frac{1}{2m}(\boldsymbol{\nabla}-\mathrm{i}eA)^2-\mu\right)\psi_\sigma\right]$$

$$\times\exp\Big[-(\gamma_1)^2\int \mathrm{d}^4x\,\varphi_1^*\varphi_1+(-\lambda_1 U)^{1/2}\gamma_1$$

$$\times\int \mathrm{d}^4x\,(\bar\psi_\uparrow\bar\psi_\downarrow\varphi_1+\psi_\downarrow\psi_\uparrow\varphi_1^*+j_1\varphi_1^*+j_1^*\varphi_1)$$

$$-\gamma_2^2\int \mathrm{d}^4x\,\varphi_2^*\varphi_2+(\lambda_2 U)^{1/2}\gamma_2$$

$$\times\int \mathrm{d}^4x\,(\bar\psi_\uparrow\psi_\downarrow\varphi_2+\bar\psi_\downarrow\psi_\uparrow\varphi_2^*+j_2\varphi_2^*+j_2^*\varphi_2)$$

$$-(\gamma_3)^2\int \mathrm{d}^4x\,\varphi_3^*\varphi_3+\left(-\frac{\lambda_3}{2}U\right)^{1/2}\gamma_3\int \mathrm{d}^4x\,[(\bar\psi_\uparrow\psi_\uparrow+\bar\psi_\downarrow\psi_\downarrow)\varphi_3$$

$$+(\bar{\psi}_\uparrow\psi_\uparrow+\bar{\psi}_\downarrow\psi_\downarrow)\varphi_3^*+j_3\varphi_3^*+\varphi_3 j_3^*]$$
$$-(\gamma_4)^2\int d^4x\,\varphi_4^*\varphi_4+\left(\frac{\lambda_4}{2}U\right)^{1/2}\gamma_4\int d^4x\,[(\bar{\psi}_\uparrow\psi_\uparrow-\bar{\psi}_\downarrow\psi_\downarrow)\varphi_4$$
$$+(\bar{\psi}_\uparrow\psi_\uparrow-\bar{\psi}_\downarrow\psi_\downarrow)\varphi_4^*+j_4\varphi_4^*+\varphi_4 j_4^*]\Big]. \tag{10}$$

Putting all sources to zero we get the partition sum

$$Z=Z[\{j_i j^*\}]_{j=j^*=0}. \tag{11}$$

For the Lagrange function in (10) given by

$$L=L_0+L_1 \tag{12}$$

we have (compare (6))

$$L_0=-\int_V d^3r\left[\sum_{\sigma=\uparrow,\downarrow}\bar{\psi}_\sigma\left(\frac{\partial}{\partial\tau}-\frac{\nabla^2}{2m}-\mu\right)\psi_\sigma+\sum_{i=1}^{4}(\gamma_1)^2\varphi_i^*\varphi_i\right] \tag{13}$$

and

$$L_1=-\int_V d^3r\left[\sum_{\sigma=\uparrow,\downarrow}\left(\frac{-ie}{2m}[\bar{\psi}_\sigma\nabla\psi_\sigma-(\nabla\bar{\psi}_\sigma)\psi_\sigma]A+\frac{e^2}{2m}\bar{\psi}_\sigma\psi_\sigma A^2\right)\right.$$
$$-(-\lambda_1 U)^{1/2}\gamma_1(\bar{\psi}_\uparrow\bar{\psi}_\downarrow\varphi_1+\psi_\downarrow\psi_\uparrow\varphi_1^*)$$
$$-(\lambda_2 U)^{1/2}\gamma_2(\bar{\psi}_\uparrow\psi_\downarrow\varphi_2+\bar{\psi}_\downarrow\psi_\uparrow\varphi_2^*)$$
$$-\left(-\frac{\lambda_3}{2}U\right)^{1/2}\gamma_3[(\bar{\psi}_\uparrow\psi_\uparrow+\bar{\psi}_\downarrow\psi_\downarrow)\varphi_3+(\bar{\psi}_\uparrow\psi_\uparrow+\bar{\psi}_\downarrow\psi_\downarrow)\varphi_3^*]$$
$$\left.-\left(\frac{\lambda_4}{2}U\right)^{1/2}\gamma_4[(\bar{\psi}_\uparrow\psi_\uparrow-\bar{\psi}_\downarrow\psi_\downarrow)\varphi_4+(\bar{\psi}_\uparrow\psi_\uparrow-\bar{\psi}_\downarrow\psi_\downarrow)\varphi_4^*]\right]. \tag{14}$$

Using the Fourier transformation

$$\psi_\sigma(r,\tau)=\frac{1}{(\beta V)^{1/2}}\sum_{kn}\psi_\sigma(k,n)\exp[i(k\cdot r-\xi_n\tau)] \tag{15}$$

(analogously, for the fields φ_i, $i=1,\ldots,4$) we get for the free action

$$\int_0^\beta d\tau\,L_0=-\sum_{kn}\left(\sum_{\sigma=\uparrow,\downarrow}\bar{\psi}_\sigma(k,n)(\omega_k-i\xi_n)\psi_\sigma(k,n)\right.$$
$$\left.+\sum_{i=1}^{4}(\gamma_i)^2\varphi_i^*(k,n)\varphi_i(k,n)\right). \tag{16}$$

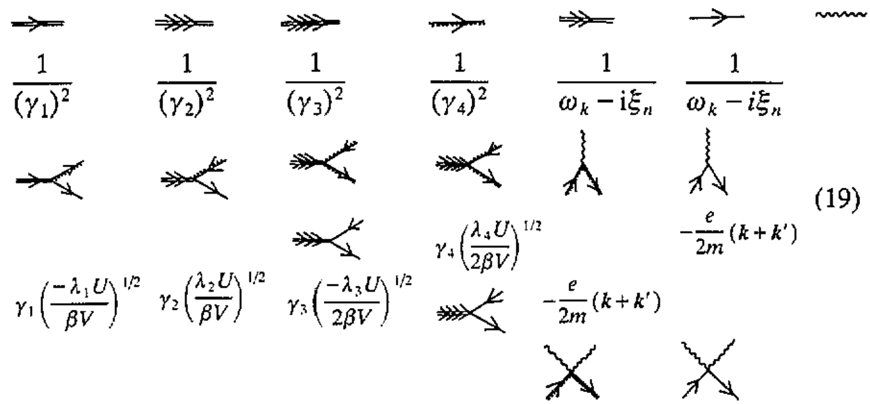
Thus the propagator of an electron with spin σ and four-momentum (k,n) is given by

$$1/(\omega_k-i\xi_n)\qquad\text{with }\ \omega_k=k^2/2m-\mu\quad\text{and}\quad\xi_n=(\pi/\beta)(2n+1),\quad n=0,\pm1,\ldots \tag{17}$$

and the propagator of the auxiliary fields is given by

$$1/(\gamma_i)^2\qquad\text{for }\ i=1,\ldots,4. \tag{18}$$

The propagators of φ_i-fields do not depend on the four-momentum because the Lagrange function (13) does not contain the corresponding kinetic energy terms. One obtains the Feynman rules by substitution of the Fourier decomposition of the fields in the interaction Lagrange function (14). We use the following notation:



	φ_1	φ_2	φ_3	φ_4	$\psi_\uparrow$	$\psi_\downarrow$	A
field							
propagator	$\frac{1}{(\gamma_1)^2}$	$\frac{1}{(\gamma_2)^2}$	$\frac{1}{(\gamma_3)^2}$	$\frac{1}{(\gamma_4)^2}$	$\frac{1}{\omega_k - \mathrm{i}\xi_n}$	$\frac{1}{\omega_k - i\xi_n}$	
vertex	$\gamma_1\left(\frac{-\lambda_1 U}{\beta V}\right)^{1/2}$	$\gamma_2\left(\frac{\lambda_2 U}{\beta V}\right)^{1/2}$	$\gamma_3\left(\frac{-\lambda_3 U}{2\beta V}\right)^{1/2}$	$\gamma_4\left(\frac{\lambda_4 U}{2\beta V}\right)^{1/2}$	$-\frac{e}{2m}(k+k')$	$-\frac{e}{2m}(k+k')$	
				$-\gamma_4\left(\frac{\lambda_4 U}{2\beta V}\right)^{1/2}$	$\frac{-e^2}{2m}$	$\frac{-e^2}{2m}$	

(19)

In this theory any connected Feynman graph consists of a single loop, on which any number of vertices can be placed. The generating functional for the connected Green functions is defined by

$$W[\{j, j^*\}] = \ln(Z[\{j, j^*\}]). \tag{20}$$

The Legendre transformation of $W[\{j, j^*\}]$ reads

$$\Gamma[\{\varphi_c, \varphi_c^*; A\}] = \int \mathrm{d}^4x \sum_{i=1}^{4} (j_i\varphi_{i_c}^* + \varphi_{i_c} j_i^*) - W[\{j, j^*\}] \tag{21}$$

where $\varphi_{i_c}(\gamma)$ means the expectation value of $\varphi_i(\gamma)$ in the presence of the sources $j_i(x)$:

$$\varphi_{i_c}(x) = \langle\varphi_i(x)\rangle_{j_i \neq 0} \tag{22}$$

and $\Gamma[\{\varphi_{c_i}\varphi_c^*; A\}]$ is the effective action. The partition sum is then

$$Z = Z[\{j, j^*\}]_{j_1 = j_1^* = \ldots = j_4^* = 0} = \exp\{-\Gamma[\{\varphi_{c_i}\varphi_c^*; A\}]\}_{j_1 = \ldots = j_4^* = 0} \tag{23}$$

and as equations for $\langle\varphi_i(x)\rangle_{j_i=0} = \varphi_{i_c}^0(x), i = 1, \ldots, 4$ we obtain generalized Ginzburg–Landau equations:

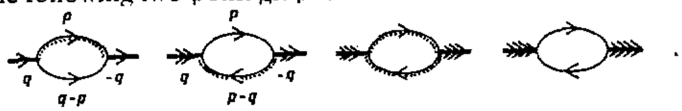
$$[\partial\Gamma/\partial\varphi_{i_c}(x)]|_{j_i = j_i^* = 0} = 0, \qquad [\partial\Gamma/\partial\varphi_{i_c}^*(x)]|_{j_i = j_i^* = 0} = 0, \qquad i = 1, \ldots, 4 \tag{24}$$

In order to calculate the Ginzburg–Landau functional $\Gamma[\{\varphi_c, \varphi_c^*; A\}]$ we consider the generating functional for the proper or one-particle irreducible diagrams

$$\kappa[\{\varphi_c, \varphi_c^*; A\}] = \int \mathrm{d}^4x \sum_{i=1}^{4} (\gamma_i)^2 |\varphi_{i_c}|^2 - \Gamma[\{\varphi_c, \varphi_c^*; A\}] \tag{25}$$

that we can find by perturbation theory. We consider all one-loop diagrams with two and four external lines and we are only interested in graphs for which the sum of external momenta is zero. (Diagrams with an odd number of φ-lines do not give contributions for temperatures $T \neq 0$.)

We compute firstly the coefficients $\tilde{V}_2^{ii}(q, -q; A = 0)$ proportional to $\varphi_{i_c}(q)\varphi_{i_c}(-q)$ in $\kappa[\{\varphi\varphi_{c_i}\varphi_c^*; A\}]$. Using (10), (17) and (19) we have with $p = (p, n)$ for $A = 0$ and $q + q' = 0$ the following two-point graphs:



(26)

The sum over the internal lines yields for all diagrams of (26) the same result:

$$\Sigma_2^q = \frac{1}{\beta V}\sum_{p,n}\left\{\frac{1}{\omega_p - \mathrm{i}\xi_n}\exp\left(\pm q\frac{\partial}{\partial(\mp p)}\right)\frac{1}{\omega_p + \mathrm{i}\xi_n}\right\} \simeq a + bq^2 + \mathrm{O}((q^4)) \tag{27}$$

with

$$a = \frac{1}{\beta V}\sum_{r,n}\frac{1}{\omega_p^2 + \xi_n^2} \tag{28}$$

and

$$b = \frac{1}{\beta V}\sum_{p,n}\left\{\frac{1}{\omega_p - \mathrm{i}\xi_n}\frac{1}{(q)^2}\frac{1}{2}\left(q\frac{\partial}{\partial p}\right)^2\frac{1}{\omega_p + \mathrm{i}\xi_n}\right\}$$
$$= \frac{1}{\beta V}\sum_{p,n}\left\{\frac{-\omega_p}{(\omega_p^2 + \xi_n^2)^2}\frac{1}{2m} + \frac{\omega_p^2 - \xi_n^2}{(\omega_p^2 + \xi_n^2)^3}\frac{1}{(q)^2}\left(\frac{p\cdot q}{m}\right)^2\right\}. \tag{29}$$

We restrict ourselves to second-order terms in q because the terms with $|q| \ll 1$ give the main contributions to the sum Σ_2^q.

The last two graphs in (26) have to be counted twice since for the fields φ_3 and φ_4 two vertices exist (compare (19)). The factor 2 cancels with the factor $\frac{1}{2}$ from the product of the vertices. Both graphs which yield the coefficient $\tilde{V}_2^{34}$ cancel each other and therefore they are not drawn in (26). Thus we obtain for the coefficients

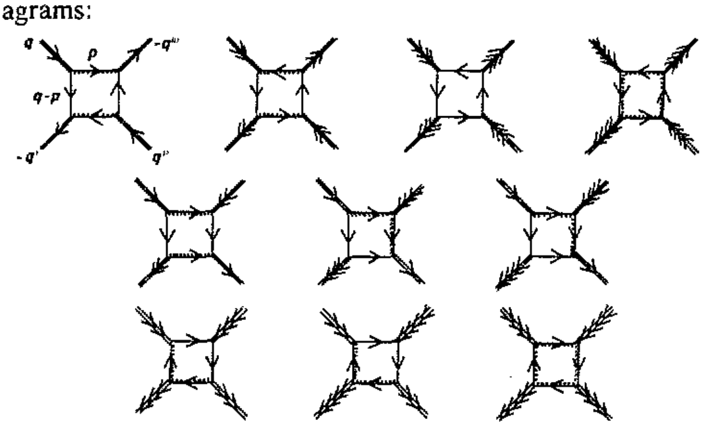
$$\tilde{V}_2^{ii}(q, -q_i; A = 0) = \mp\lambda_i U(\gamma_i)^2(a + b(q)^2) \tag{30}$$

where the upper sign is valid for $i = 1, 3$ and the lower sign for $i = 2, 4$. The coefficients $V_2^{ii}(q, -q; A = 0)$ at $|\varphi_{i_c}(q)|^2$ in the effective action are

$$V_2^{ii}(q, -q; A = 0) = (\gamma_i)^2[1 \pm \lambda_i U(a + b(q)^2)] \tag{31}$$

with the same convention over the signs as in equation (30).

Secondly, we consider the four-point terms in $\Gamma[\{\varphi_c, \varphi_c^*; A\}]$ for $A = 0$. We have the following diagrams:



(32)

The graphs in the upper line of (32) represent the coefficients $V_4^{iiii}(\boldsymbol{q}, -\boldsymbol{q}', \boldsymbol{q}'', -\boldsymbol{q} + \boldsymbol{q}' - \boldsymbol{q}'')$ at $\varphi_{i_c}(\boldsymbol{q})\varphi_{i_c}(-\boldsymbol{q}')\varphi_{i_c}(\boldsymbol{q}'')\varphi_{i_c}(-\boldsymbol{q} + \boldsymbol{q}' - \boldsymbol{q}'')$ in $\Gamma[\{\varphi_c, \varphi_c^*; A = 0\}]$. From these we obtain, e.g., the coefficient at $(\varphi_{1c}\varphi_{1c}^*)^2$ putting $\boldsymbol{q}' = \boldsymbol{q}'' = \boldsymbol{q}$. The last two graphs in the upper line of (32) are counted twice again, and similarly the graph which yields the coefficient V_4^{3434} because two vertices belong to the fields φ_3 and φ_4, and the summation over all internal lines gives for all four-point diagrams (32) the same result, as according to (16) and (17) the propagator of an uncorrelated electron does not depend on the spin and $\omega_p = (p^2/2m) - \mu$ is a function of $|\boldsymbol{p}|^2$. In the second and third lines of (32) are represented the coefficients $V_4^{\cdots}(\boldsymbol{q}, -\boldsymbol{q}', -\boldsymbol{q}'', -\boldsymbol{q} + \boldsymbol{q}' + \boldsymbol{q}'')$ which describe the coupling between the different auxiliary fields. The graphs which the coefficients V_4^{1413}, V_4^{1314} and V_4^{2423}, V_4^{2324} yield cancel pairwise and therefore they are not drawn in (32). Using (19) we obtain:

$$V_4^{iiii}(\boldsymbol{q}, -\boldsymbol{q}', \boldsymbol{q}'', -\boldsymbol{q} + \boldsymbol{q}' - \boldsymbol{q}'') = \begin{cases} (\lambda_i U)^2 (\gamma_i)^4 \Sigma_4^{\boldsymbol{q}, -\boldsymbol{q}', \boldsymbol{q}''} & i = 1, 2 \\ \tfrac{1}{2}(\lambda_i U)^2 (\gamma_i)^4 \Sigma_4^{\boldsymbol{q}, -\boldsymbol{q}', \boldsymbol{q}''} & i = 3, 4 \end{cases}$$

$$V_4^{1212}(\boldsymbol{q}, -\boldsymbol{q}', -\boldsymbol{q}'', -\boldsymbol{q} + \boldsymbol{q}' + \boldsymbol{q}'') = -\lambda_1 \lambda_2 U^2 (\gamma_1)^2 (\gamma_2)^2 \Sigma_4^{\boldsymbol{q}, -\boldsymbol{q}', -\boldsymbol{q}''}$$

$$V_4^{i3i3}(\boldsymbol{q}, -\boldsymbol{q}', -\boldsymbol{q}'', -\boldsymbol{q} + \boldsymbol{q}' + \boldsymbol{q}'') = \pm \tfrac{1}{2}\lambda_i \lambda_3 U^2 (\gamma_i)^2 (\gamma_3)^2 \Sigma_4^{\boldsymbol{q}, -\boldsymbol{q}', -\boldsymbol{q}''} \qquad i = \begin{cases} 1 \\ 2 \end{cases} \tag{33}$$

$$V_4^{i4i4}(\boldsymbol{q}, -\boldsymbol{q}', -\boldsymbol{q}'', -\boldsymbol{q} + \boldsymbol{q}' + \boldsymbol{q}'') = \pm \tfrac{1}{2}\lambda_i \lambda_4 U^2 (\gamma_i)^2 (\gamma_4)^2 \Sigma_4^{\boldsymbol{q}, -\boldsymbol{q}', -\boldsymbol{q}''} \qquad i = \begin{cases} 1 \\ 2, 3 \end{cases}$$

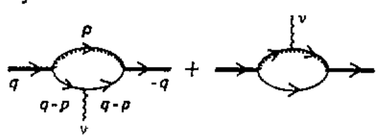
with $\Sigma_{i=1}^4 \lambda_i = 1$ (compare (7)) and

$$\Sigma_4^{\boldsymbol{q}, -\boldsymbol{q}', \boldsymbol{q}''} = \frac{1}{\beta V} \sum_{p,n} \left\{ \frac{1}{\omega_p - \mathrm{i}\xi_n} \exp\left(\boldsymbol{q} \frac{\partial}{\partial \boldsymbol{p}} \right) \frac{1}{\omega_p + \mathrm{i}\xi_n} \right.$$
$$\left. \times \exp\left((\boldsymbol{q} - \boldsymbol{q}') \frac{\partial}{\partial \boldsymbol{p}} \right) \frac{1}{\omega_p - \mathrm{i}\xi_n} \exp\left((\boldsymbol{q}' - \boldsymbol{q}' + \boldsymbol{q}'') \frac{\partial}{\partial \boldsymbol{p}} \right) \frac{1}{\omega_p + \mathrm{i}\xi_n} \right\}. \tag{34}$$

Obviously, the sum Σ_4 yields the main contributions for $\boldsymbol{q}' = \boldsymbol{q}' = \boldsymbol{q}''$. Thirdly, in order to write down the complete gauge-invariant action in our approximation we still have to calculate the $\varphi_{i_c}\varphi_{i_c}^* A$ and $\varphi_{i_c}\varphi_{i_c}^* A^2$ graphs.

One can obtain the results from (26) by a minimum substitution or by straightforward calculation using (19). The coefficient $V_2^{11}(\boldsymbol{q}, -\boldsymbol{q}; A_\nu)$, $\nu = 1, 2, 3$ at $\varphi_{1_c}(\boldsymbol{q})\varphi_{1_c}^*(\boldsymbol{q})A_\nu$, for example, is given by

$V_2^{11}(\boldsymbol{q}_1 - \boldsymbol{q}; A_\nu) =$



$$= \frac{K_1}{\beta V} \sum_p \left(\frac{1}{\omega_p - \mathrm{i}\xi_n} \frac{1}{(\omega_{q-p} + \mathrm{i}\xi_n)^2} (q_\nu - p_\nu) \right.$$
$$\left. + \frac{1}{(\omega_p - \mathrm{i}\xi_n)^2} \frac{1}{(\omega_{q-p} + \mathrm{i}\xi_n)} p_\nu \right). \tag{35}$$

By restriction on terms linear in q we get

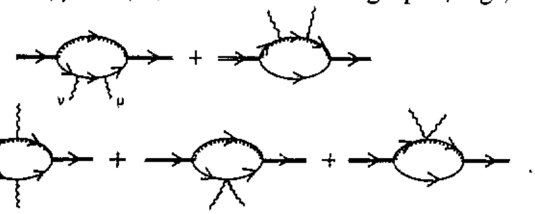
$$V_2^{ii}(\boldsymbol{q},-\boldsymbol{q};A_\nu)=\frac{K_i}{\beta V}\sum_{p,n}\left\{\pm\frac{\omega_p}{(\omega_p^2+\zeta_n^2)^2}q_\nu+\frac{1}{(\omega_p^2+\xi_n^2)^2}\left(\frac{\boldsymbol{q}\cdot\boldsymbol{p}}{m}\right)p_\nu\right.$$
$$\left.\mp 2\frac{\omega_p^2-\xi_n^2}{(\omega_p^2+\xi_n^2)^3}\left(\frac{\boldsymbol{q}\cdot\boldsymbol{p}}{m}\right)p_\nu+\mathrm{O}((q^3))\right\}\qquad \nu=1,2,3 \tag{36}$$

with the upper sign for $i=1,3$ and the lower sign for $i=2,4$. The constants K_i are abbreviations:

$$K_i=\pm\lambda_i U(\gamma_i)^2\,(l/m)\qquad i=1,\ldots,4, \tag{37}$$

where the minus sign is valid for $i=2,4$. The coefficients $V_2^{ii}(\boldsymbol{q},-\boldsymbol{q};A_\nu,A_\mu)$ at $\varphi_{i_c}(\boldsymbol{q})\varphi_{i_c}^*(\boldsymbol{q})A_\nu A_\mu$ with $\nu,\mu=1,2,3$ result from five graphs, e.g.,

$V_2^{11}(\boldsymbol{q}_1-\boldsymbol{q};A_\nu,A_\mu)=$



(38)

We restrict ourselves to graphs where all external momenta are equal to zero. Using (19) it follows that

$$V_2^{ii}(0,0;A_\nu,A_\mu)=-K_i\frac{e}{m}\frac{1}{\beta V}\sum_{p,n}\left\{\frac{\omega_p^2-\xi_n^2}{(\omega_p^2+\xi_n^2)^3}2p_\nu p_\mu\right.$$
$$\left.\mp\frac{1}{(\omega_p^2+\xi_n^2)^2}p_\nu p_\mu-\frac{\omega_p}{(\omega_p^2+\xi_n^2)^3}m\delta_{\nu\mu}\right\}. \tag{39}$$

Now we can write down the complete gauge-invariant effective action

$$\Gamma[\{\varphi_c,\varphi_c^*;A\}]=\sum_{i=1}^{4}\left(V_2^{ii}(\boldsymbol{q},-\boldsymbol{q};A=0)+\sum_{\nu=1}^{3}V_2^{ii}(\boldsymbol{q},-\boldsymbol{q};A_\nu)A_\nu\right.$$
$$\left.+\sum_{\nu,\mu=1}^{3}V_2^{ii}(0,0;A_\nu,A_\mu)A_\nu A_\mu\right)|\varphi_{i_c}(\boldsymbol{q})|^2$$
$$+\sum_{i=1}^{4}V_4^{iiii}(0,0,0,0)(|\varphi_{i_c}(\boldsymbol{q})\varphi_{i_c}(-\boldsymbol{q}))^2$$
$$+V_4^{1212}(0,0,0,0)|\varphi_{1_c}(\boldsymbol{q})|^2|\varphi_{2_c}(\boldsymbol{q})|^2$$
$$+\sum_{i=1}^{2}V_4^{i3i3}|\varphi_{i_c}(\boldsymbol{q})|^2\varphi_{3_c}(-\boldsymbol{q})\varphi_{3_c}(\boldsymbol{q})$$
$$+\sum_{i=1}^{3}V_4^{i4i4}\varphi_{i_c}(\boldsymbol{q})\varphi_{i_c}(-\boldsymbol{q})\varphi_{4_c}(\boldsymbol{q})\varphi_{4_c}(-\boldsymbol{q})-\tfrac{1}{2}[\boldsymbol{q}\times\boldsymbol{A}]^2. \tag{40}$$

The coefficients in (40) are given by (31), (36), (39) and (33) whereby in (33) all external moments and the electromagnetic field are taken to be equal to zero. The restriction on $|\boldsymbol{q}|\ll 1$ means that here we regard long-range order and ferromagnetic ($\boldsymbol{q}=\boldsymbol{0}$) or paramagnetic solutions only.

The condition for the minimum effective action yields generalized Ginzburg–Landau equations according to (24).

4. Discussion

The investigations of the Ginzburg–Landau functional (40) and of the generalized Ginzburg–Landau equations allow discussion of the phase transitions of the Hubbard model (3). The φ-fields play the role of order parameters. Since the s-band Hubbard model shows particle–hole symmetry (see, e.g., [2]) the considerations can be restricted to band fillings $0 \leqslant n \leqslant 1$, i.e. for the order parameters φ_{i_c} (compare (22)) we have $|\varphi_{i_c}|^2 \leqslant 1, i = 1, \ldots, 4$.

As an example we consider $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$ in equation (7) and use an attractive coupling constant—this means that we put $U = -g$ with $-g > 0$. Of course, we obtain the well known results of the BCS theory [6].

From the vanishing of the coefficient $V_2^{11}(\boldsymbol{q}, -\boldsymbol{q}; A = 0)$, for $q \to 0$ the transition temperature T_{1_c} follows in the usual manner. One introduces the density of states

$$\frac{1}{V}\sum_{p} = \frac{1}{(2\pi)^d}\int \mathrm{d}^d \boldsymbol{p} = \int_{-\infty}^{\infty} \mathrm{d}\omega\, \rho(\omega)$$

where d is the dimension of the problem and get at the Fermi surface $\omega = p^2/2m - \mu = 0$ using (31), (28) and (17):

$$a = \rho(0)\frac{1}{\beta}\sum_{n}\frac{\pi}{\xi_n} = 2\rho(0)\sum_{n=0}^{\infty}\frac{1}{2n+1}. \tag{41}$$

Since one assumes in the BCS theory that the attractive coupling constant g is caused by interaction with phonons, one usually introduces a cut-off $n_{\max}$ in the sum over n equivalent to a cut-off in the phonon energy spectrum at the Debye energy $\omega_{\mathrm{D}} = k_{\mathrm{B}}T\pi(2n_{\max} + 1)$ and then it follows that

$$a \approx 2\rho(0)\sum_{n=0}^{n_{\max}}\frac{1}{2n+1} = \rho(0)\ln\left(\frac{\omega_{\mathrm{D}}}{k_{\mathrm{B}}T\pi}4\gamma\right) \tag{42}$$

where $\ln\gamma = C = 0.577$ is the Euler constant. For b (compare (29)), for example, one obtains at the Fermi surface

$$b \approx \frac{1}{24}\frac{1}{m}\rho(0)\mu\zeta(3, \tfrac{1}{2})\frac{1}{(K_{\mathrm{B}}T\pi)^2} \tag{43}$$

where $\zeta(3, \frac{1}{2}) = \Sigma_{n=0}^{\infty}(n + \frac{1}{2})^{-3}$ is the Riemann zeta function and where

$$\int \mathrm{d}\Omega_p\, p_\nu p_\mu = \delta_{\nu\mu}\tfrac{1}{3}\boldsymbol{p}^2. \tag{44}$$

The coefficient at $|\varphi_{1_c}|^2$ in the Ginzburg–Landau functional vanishes according to the theory of phase transitions at the critical temperature (see, e.g., [8]). One sees that, regarding (31), in order to find the critical temperature one needs a cut-off in the energy spectrum. It yields

$$T_{1_c} = 4\gamma\omega_{\mathrm{D}}/\pi k_{\mathrm{B}}\exp[-1/g\rho(0)] \tag{45}$$

i.e. we obtain the well known formula of the BCS theory. We remark that this physical result is still independent of the constants γ_i as expected.

With the critical temperature T_{1_c} one can write for the coefficient at $|\varphi_{1_c}|^2$ in the effective action

$$V_2^{11}(0,0;\boldsymbol{A}=\boldsymbol{0}) = (\gamma_1)^2 g\rho(0)\ln(T/T_{1_c}). \tag{46}$$

The coefficients $V_4^{\cdots}$ in the effective action (40) are at the Fermi surface for all combinations of λ_i with $\Sigma_{i=1}^4 \lambda_i = 1$ proportional to

$$\frac{1}{\beta}\sum_n \rho(0)\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{(\omega^2+\xi_n^2)^2} = \tfrac{1}{8}\,\rho(0)\,\zeta(3,\tfrac{1}{2})\,\frac{1}{(k_B T\pi)^2} \tag{47}$$

and for the quantities (36) and (39) it yields in the same approximation

$$V_2^{ii}(\boldsymbol{q},-\boldsymbol{q};A_\nu) = \begin{cases}\lambda_1 U(\gamma_1)^2(e/m)cq_\nu & \text{for } i=1\\ 0 & \text{for } i=2,3,4\end{cases} \tag{48}$$

and

$$V_2^{ii}(\boldsymbol{q},-\boldsymbol{q};A_\nu,A_\mu) = \begin{cases}\lambda_1 U(\gamma_1)^2(e^2/m)c\delta_{\nu\mu} & \text{for } i=1\\ 0 & \text{for } i=2,3,4\end{cases} \tag{49}$$

with

$$c = \tfrac{1}{8}\mu\rho(0)\zeta(3,\tfrac{1}{2})[1/(k_B T\pi)^2]. \tag{50}$$

Equations (48) and (49) show that near the Fermi surface and for $|\boldsymbol{q}| \ll 1$ the electromagnetic field is coupled at the auxiliary field only, which describes a particle of mass $2m$ and charge $2e$ (Cooper pair), and the coefficient at $\varphi_{1_c}\varphi_{1_c}^*$ for $\boldsymbol{A} \neq \boldsymbol{0}$ reads $\lambda_1 U(\gamma_1)^2 c(1/4m)(\boldsymbol{q}-2e\boldsymbol{A})^2$. The Ginzburg–Landau equations (24) form for this special case a coupled differential system for the φ-fields, too.

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